Ergodic Properties of Markov Processes

Exercises for week 10

Exercise 1 Show that it indeed suffices to consider the case $\mathbf{E}(f \mid \mathscr{I}) = 0$ in the proof of Birkhoff's ergodic theorem.

Exercise 2 Show that if a transition operator T has an invariant measure of the form δ_x for some $x \in \mathcal{X}$, then this invariant measure is automatically ergodic.

Exercise 3 Given a probability measure $\mathbf{P}$ on $\mathcal{X}$, we introduce an equivalence relation on the subsets of $\mathcal{X}$ by $A \sim B$ if there exists $C \subset \mathcal{X}$ with $\mathbf{P}(C) = 0$ such that $(A \setminus B) \cup (B \setminus A) \subset C$. Show that, given a set A and a countable family of sets $\{A_n\}$ such that $A_n \sim A$ for every n, one also has $A \sim \bigcup A_n$ and $A \sim \bigcap A_n$.

Exercise 4 Show that if the map $x \mapsto P(x, \cdot)$ is continuous when the space of probability measures is endowed with the total variation distance, then this transition probability is strong Feller.

Show that if the map $x \mapsto P(x, \cdot)$ Lipschitz continuous with Lipschitz constant K, then $T\varphi$ is Lipschitz continuous with Lipschitz constant $K \sup_x |\varphi(x)|$ for every bounded measurable function $\varphi\colon \mathcal{X} \to \mathbf{R}$.

Exercise 5 Let P be some Markov transition probabilities on a Polish space $\mathcal{X}$ which is strong Feller (i.e. the corresponding transition operator T maps bounded measurable functions into bounded continuous functions). Show that if π is invariant for P and $\varphi\colon \mathcal{X} \to \mathbf{R}$ is a function such that $\varphi(x) = 1$ on a set of π-measure 1, then $(T\varphi)(x) = 1$ for every x in the topological support of π. (Recall that the topological support of π is the smallest closed set of measure 1. Such a set always exists in Polish spaces.)

* **Exercise 6** Using the previous exercise, show that any two distinct ergodic invariant measures for a strong Feller transition probability P have disjoint topological supports.

Conclude that if P is strong Feller and there exists a point $x \in \mathcal{X}$ such that $P(y, A) > 0$ for every $y \in \mathcal{X}$ and for every neighbourhood of x, then P can have at most one invariant measure.