Ergodic properties of highly degenerate 2D stochastic Navier-Stokes equations

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Abstract

This note presents the results from "Ergodicity of the degenerate stochastic 2D Navier-Stokes equation" by M. Hairer and J. C. Mattingly. We study the Navier-Stokes equation on the two-dimensional torus when forced by a finite-dimensional Gaussian white noise and give conditions under which the system is ergodic. In particular, our results hold for specific choices of four-dimensional Gaussian white noise.

Résumé

Cette Note présente les résultats de l'article "Ergodicity of the degenerate stochastic 2D Navier-Stokes equation" par M. Hairer et J. C. Mattingly. Nous étudions l'équation de Navier-Stokes sur le tore bidimensionel, excitée par un bruit blanc gaussien de dimension finie. Nous donnons des conditions suffisantes pour que la solution soit ergodique. Nos résultats sont en particulier vrais dans certains cas de bruit blanc gaussien de dimension quatre.

1. Introduction

This note reports on recent progress made in [12] on the study of the two dimensional Navier-Stokes equation driven by an additive stochastic forcing. These results make use in a critical fashion of another set of recent results from Mattingly and Pardoux [20]. Recall that the Navier-Stokes equation describes the time evolution of an incompressible fluid. We quickly recall the formulation of the Navier-Stokes equation, referring the reader to [13] and the two source articles for more information. In vorticity form, the Navier-Stokes equation is given by

$$\begin{cases} \dfrac{\partial w}{\partial t}(t,x) + B(w,w)(t,x) = \nu \Delta w(t,x) + \dfrac{\partial W}{\partial t}(t,x) \\ w(0,x) = w_0(x), \end{cases} \tag{1}$$

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where $x = (x_1, x_2) \in \mathbb{T}^2$, the two-dimensional torus $[0, 2\pi] \times [0, 2\pi]$, $\nu > 0$ is the viscosity constant, $\frac{\partial W}{\partial t}$ is a white-in-time stochastic forcing to be specified below, and $B(w, \tilde{w})(x) = \sum_{i=1}^{2} (\mathcal{K}w)_i(x) \frac{\partial \tilde{w}}{\partial x_i}(x)$, where $\mathcal{K}$ is the Biot-Savart integral operator which reconstructs the velocity from the vorticity. Its definition is given in [13] in the sequence. As in [13], we define a convenient basis in which we will perform all explicit calculations. Setting $\mathbb{Z}^2_+ = \{(j_1, j_2) \in \mathbb{Z}^2 : j_2 > 0\} \cup \{(j_1, j_2) \in \mathbb{Z}^2 : j_1 > 0, j_2 = 0\}$, $\mathbb{Z}^2_- = -\mathbb{Z}^2_+$ and $\mathbb{Z}^2_0 = \mathbb{Z}^2_+ \cup \mathbb{Z}^2_-$, we define a real Fourier basis for functions on $\mathbb{T}^2$ with zero spatial mean by $e_k(x)$ equals $\sin(k \cdot x)$ if $k \in \mathbb{Z}^2_+$ and $\cos(k \cdot x)$ if $k \in \mathbb{Z}^2_-$. Write $w(t, x) = \sum_{k \in \mathbb{Z}^2_0} \alpha_k(t) e_k(x)$ for the expansion of the solution in this basis. We solve (1) on the space $\mathbb{L}^2 = \{f = \sum_{k \in \mathbb{Z}^2_0} a_k e_k : \sum |a_k|^2 < \infty\}$. For $f = \sum_{k \in \mathbb{Z}^2_0} a_k e_k$, we define the norms $\|f\|^2 = \sum |a_k|^2$ and $\|f\|_1^2 = \sum |k|^2 |a_k|^2$.

The emphasis of this note will be on forcing which directly excites only a few degrees of freedom. Such forcing is both of primary modeling interest and is technically the most difficult. Specifically we consider forcing of the form $W(t, x) = \sum_{k \in \mathcal{Z}_*} \sigma_k W_k(t) e_k(x)$. Here $\mathcal{Z}_*$ is a finite subset of $\mathbb{Z}^2_0$, $\sigma_k > 0$, and $\{W_k : k \in \mathcal{Z}_*\}$ is a collection of mutually independent standard scalar Brownian Motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. As described in [13], the spread of the randomness through the system is captured by the sets $\mathcal{Z}_0 = \mathcal{Z}_* \cap (-\mathcal{Z}_*)$, $\mathcal{Z}_n = \left\{ \ell + j \in \mathbb{Z}^2_0 \;:\; j \in \mathcal{Z}_0,\ \ell \in \mathcal{Z}_{n-1} \text{ with } \ell^\perp \cdot j \neq 0,\ |j| \neq |\ell| \right\}$, and lastly, $\mathcal{Z}_\infty = \bigcup_{n=1}^{\infty} \mathcal{Z}_n$. $\mathcal{Z}_\infty$ captures the directions to which the randomness has spread. And as discussed in [13], its structure is related to the formal commutators of the infinite dimensional diffusion on $\mathbb{L}^2$ formally associated to (1).

The note [13] contains the results from Mattingly and Pardoux [20] on Malliavin calculus applied to (1) giving control of the smoothing of the probability transition density in the directions contained in $\operatorname{span}\{e_k : k \in \mathcal{Z}_\infty\}$. It also contains the following characterization of the case $\mathcal{Z}_\infty = \mathbb{Z}^2_0$ from Hairer and Mattingly [12].

Proposition 1.1 *One has $\mathcal{Z}_\infty = \mathbb{Z}^2_0$ if and only if integer linear combinations of elements of $\mathcal{Z}_0$ generate $\mathbb{Z}^2_0$ and there exist at least two elements in $\mathcal{Z}_0$ with unequal euclidean norm.*

See [13] for further discussion of this proposition and related issues.

This note describes results obtained in Hairer and Mattingly [12], which uses the tools from [20] to build a theory which, when applied to (1), proves that it has a unique invariant measure under extremely general and essentially sharp assumptions. In addition to the tools from [20], they introduce a new concept and tool which together provide an abstract framework in which the ergodicity of (1) is proven. The concept is a generalization of the strong Feller property for a Markov process which, for reasons that will be made clear below, is called the *asymptotic strong Feller* property. The main feature of this property is that a diffusion which is irreducible and asymptotically strong Feller can have at most one invariant measure. It thus yields a natural generalization of Doob's theorem. The tool is an approximate integration by parts formula, in the sense of Malliavin calculus, which is used to prove that the system enjoys the asymptotic strong Feller property. To the best of the authors knowledge, this paper is the first to prove ergodicity of a nonlinear stochastic partial differential equation (SPDE) under assumptions comparable to those assumed when studying finite dimensional stochastic differential equations.

The ergodic theory of infinite dimensional stochastic systems, and SPDEs specifically, has been a topic of intense study over the last two decades. Until recently, the forcing was always assumed to be elliptic and spatially rough. In our context this translates to $\mathcal{Z}_* = \mathbb{Z}^2_0$ and $|\sigma_k| \sim |k|^{-\alpha}$ for some positive α. Flandoli and Maslowski [9] first proved ergodic results for (1) under such assumptions. This line of inquiry was extended and simplified in [7,10]. They represent a larger body of literature which characterizes the extent to which classical ideas developed for finite dimensional Markov processes apply to infinite dimensional processes. Principally they use tools from infinite dimensional stochastic analysis to prove that the processes are strong Feller in an appropriate topology and then deduce ergodicity.

Next three groups of authors in [14,1,5], contemporaneously greatly expanded the cases known to be ergodic. They use a Foias-Prodi type reduction, first adapted to the stochastic setting in [16] and the pathwise contraction of the high spatial frequencies already used in [17] to prove ergodicity of (1) at sufficiently high viscosity. All of the results hinged on the observation that if all of the unstable directions are stochastically perturbed, then the system could be shown to be ergodic. A general overview of these ideas with simple examples can be found in [19]. These ideas have been continued and developed further in a number of papers. See for instance [4,2,18,11,15,19].

Unfortunately, the best current estimates on the number of unstable directions in (1) grow inversely with the viscosity ν. Hence the physically important limit of $\nu \to 0$ while a fixed, finite scale is forced were previously outside the scope of the theory. However there existed strong indications that ergodicity held in this case. Specifically in [6] it was shown that the generator of the diffusion associated to finite dimensional Galerkin approximations of (1) was hypoelliptic in the sense of Hörmander when only a few directions were forced. This hypoellipticity is the crucial ingredient in the proof of ergodicity from [6].

The "correct" ergodic theorem needs to incorporate in its statement information on how the randomness spreads from the few forced directions to all of the unstable directions. This understanding when combined with what had been learned in [16,17,14,1,5] should yield unique ergodicity. This is the programme executed in the papers discussed in this note.

2. Unique Ergodicity

Recall that an *invariant measure* for (1) is a probability measure $\mu_\star$ on $\mathbb{L}^2$ such that $P_t^* \mu_\star = \mu_\star$, where P_t^* is the semigroup on measures dual to the Markov transition semigroup P_t defined by $(P_t\phi)(w) = \mathbb{E}_w \phi(w_t)$ with $\phi \in C_b(\mathbb{L}^2)$. While the existence of an invariant measure for (1) can be proved by "soft" techniques using the regularizing and dissipativity properties of the flow [3,8], showing its uniqueness is a more challenging problem that requires a detailed analysis of the nonlinearity. The importance of showing the uniqueness of $\mu_\star$ is illustrated by the fact that it implies that $\frac{1}{T}\mathbb{E}\int_0^T \phi(w_t)\,dt \to \int_{\mathbb{L}^2} \phi(w)\,\mu_\star(dw)$ as $T \to \infty$, for all bounded continuous functions ϕ and all initial conditions $w_0 \in \mathbb{L}^2$. It thus gives some mathematical ground to the *ergodic assumption* usually made in the physics literature when discussing the qualitative behavior of (1). The following theorem is the main result of [12].

Theorem 2.1 *If $\mathcal{Z}_\infty = \mathbb{Z}_0^2$, then* (1) *has a unique invariant measure in* $\mathbb{L}^2$.

When combined with Proposition 1.1, this theorem gives easy to verify conditions guaranteeing a unique invariant measure.

The concept of a strong Feller Markov process appears to be less useful in infinite dimensions than in finite dimensions. In particular if P_t is strong Feller and irreducible, then the measures $P_t(u,\cdot)$ and $P_t(v,\cdot)$ are equivalent for all initial conditions $u, v \in \mathbb{L}^2$. It is easy to construct an ergodic SPDE which does not satisfy this property. Recall the following standard sufficient criteria for P_t to be strong Feller : there exists a locally bounded function $C(w,t)$ such that

$$|\nabla(P_t\phi)(w)| \le C(w,t)\|\phi\|_\infty$$

for all Fréchet differentiable functions $\phi : \mathbb{L}^2 \to \mathbb{R}$. While we will not give the exact definition of the asymptotic strong Feller property here, the following similar condition implies that the process is asymptotically strong Feller : there exists a locally bounded $C(w)$, a non-decreasing sequence of times t_n, and a strictly decreasing sequence ϵ_n with $\epsilon_n \to 0$ so that

$$|\nabla(P_{t_n}\phi)(w)| \le C(w)\|\phi\|_\infty + \epsilon_n\|\nabla\phi\|_\infty \tag{2}$$

for all Fréchet differentiable functions $\phi : \mathbb{L}^2 \to \mathbb{R}$ and all $n \ge 1$. In applications one typically has $t_n \to \infty$. Hence, the process behaves as if it acquired the strong Feller property at time infinity, which justifies the term asymptotic strong Feller.

First observe that the chain rule implies that $\langle \nabla_w (P_t\phi)(w), \xi\rangle = \mathbb{E}_w (\nabla\phi)(w_t) J_{0,t}\xi$, where $J_{0,t}$ denotes the Jacobian for the solution flow between times 0 and t. Next we seek a direction v in the Cameron-Martin space so that if $\mathcal{D}^v$ denotes the Malliavin derivative in the direction v then $J_{0,t}\xi = \mathcal{D}^v w_t$. In finite dimensions, we can often do this exactly ; however, in infinite dimensions we only know how to achieve this up to some error. Setting $\rho_t = J_{0,t}\xi - \mathcal{D}^v w_t$, we have the approximate integration by parts formula.

$$\mathbb{E}_w (\nabla\phi)(w_t) J_{0,t}\xi = \mathbb{E}_w \mathcal{D}^v[\phi(w_t)] + \mathbb{E}_w(\nabla\phi)(w_t)\rho_t = \mathbb{E}_w \phi(w_t)\int_0^t v_s dW_s + \mathbb{E}_w(\nabla\phi)(w_t)\rho_t \ .$$

From this equality one can quickly deduce (2), provided $\mathbb{E}|\int_0^\infty v_s dW_s| < \infty$ and $\mathbb{E}|\rho_t| \to 0$ as $t \to \infty$. In [12], a v_t is chosen so that these conditions hold. The analysis is complicated by the fact that the v_t constructed there is not adapted to the Brownian filtration. This complication seems unavoidable. Hence, the stochastic integral is a Skorohod integral and all of the calculations are made more complicated. Specifically, on the intervals of the form $[n, n+\frac{1}{2}]$ with $n \geq 0$ in $\mathbb{N}$, we take v to be the least squares solution to minimizing $\|\rho_{n+\frac{1}{2}}\|^2$ where the norm of v is measured in metric induced by a regularized version of the inverse of the Malliavin matrix. The matrix is regularized by adding a small multiple of the identity to make the Malliavin matrix invertible in $\mathbb{L}^2$. This has the effect of choosing v on $[n, n+\frac{1}{2}]$ to cancel the large scale components of $\rho_{n+\frac{1}{2}}$. The small scale components are handled by setting v equal to zero on time intervals of the form $(n+\frac{1}{2}, n+1)$ and letting the contractive nature at fine scales of the linearized dynamics drive them to zero. The mixing of this pathwise contractivity in the high modes modes and the probabilistic smoothing in the lower modes has its origins in the Gibbsian reductions and High/Low splitting of [16,14,1,5]. Theorem 3.2 from [13], whose proof is found in [20], is critical to ensure that cancelling the large scales of ρ with the variation v does not increase the small scales of ρ substantially. The ideas developed here can also be used to prove exponential mixing using the ideas from [18,11]. These results will be presented elsewhere.

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