Ergodic theory for infinite-dimensional stochastic processes

Martin Hairer

The aim of this note is to provide a very short overview of a number of recent results that aim at supplementing the theory of Harris chains [MT93] by an alternative theory yielding weak instead of strong convergence results. This turns out to be much more suitable in many infinite-dimensional situations where strong convergence simply doesn't take place.

Let us first recall the basic concepts of the theory of Harris chains. Throughout all of this note, we fix a Polish space $\mathcal{X}$ (the 'state space' of our system) and a stochastically continuous semigroup of Feller Markov transition kernels $\{\mathcal{P}_t\}_{t\geq 0}$ on $\mathcal{X}$. One of the most basic notions in the theory of Harris chains is that of a small set:

Definition 0.1. *A set $A \subset \mathcal{X}$ is 'small' (for the semigroup $\{\mathcal{P}_t\}_{t\geq 0}$) if there exists $\delta > 0$ and $t \geq 0$ such that $\|\mathcal{P}_t(x,\cdot\,) - \mathcal{P}_t(y,\cdot\,)\|_{\mathrm{TV}} \leq 2 - \delta$ for every pair $x, y \in A$.*

This is actually slight weakening of the usual notion of smallness (see [MT93]) which is somewhat more suitable for the purpose of this note. Another important notion is that of a *strong Feller* semigroup:

Definition 0.2. *The semigroup $\{\mathcal{P}_t\}$ is strong Feller if there exists $t \geq 0$ such that $\mathcal{P}_t\phi$ is continuous for every bounded Borel measurable function $\phi\colon \mathcal{X} \to \mathbf{R}$.*

Finally, we say that a point $x \in \mathcal{X}$ is *accessible* if there exists $t \geq 0$ such that $\mathcal{P}_t(y, U) > 0$ for every $y \in \mathcal{X}$ and every neighbourhood U of x. With these definitions at hand, we have [MT93, DPZ96]:

Theorem 0.3. *If a Markov semigroup is strong Feller and has an accessible point, then every compact set is small. Furthermore, it can have at most one invariant probability measure.*

If one knows furthermore that the process remains in 'small' regions of the phase space for most of the time, then one can say much more. In order to measure the stability of a process, the concept of a Lyapunov function is useful:

Definition 0.4. *A measurable function $V\colon \mathcal{X} \to \mathbf{R}_+$ is a Lyapunov function if there exist strictly positive constants C, γ and K such that the bound*

$$\mathcal{P}_t V(x) \leq C e^{-\gamma t} V(x) + K\;,$$

holds for every $x \in \mathcal{X}$ and for every $t \geq 0$.

Given a Lyapunov function V, one can define a weighted total variation norm on the space of all signed measures that integrate V by

$$\|\mu\|_V = \int_{\mathcal{X}} \big(1 + V(x)\big)\,|\mu(dx)|\;.$$

With this definition at hand, we have (see [MT93]; see also [HM08b] for a simple proof):

Theorem 0.5 (Harris)**.** *If a Markov semigroup admits a Lyapunov function V such that the level sets $\{x : V(x) \leq C\}$ are all small, then there exist constants $\tilde{C}$ and $\tilde{\gamma}$ such that*

$$\|\mathcal{P}_t\mu - \mathcal{P}_t\nu\|_V \leq \tilde{C}e^{-\tilde{\gamma}t}\|\mu - \nu\|_V \;,$$

for every $t \geq 0$ and every pair of probability measures μ and ν on $\mathcal{X}$. In particular, $\mathcal{P}_t$ admits exactly one invariant probability measure.

The problem with the theory of Harris chains is that it is not very well adapted to infinite-dimensional problems. While probability measures on finite-dimensional spaces are 'often' absolutely continuous with respect to some reference measure (usually Lebesgue measure), this is not so often the case in infinite dimensions due to the lack of a 'natural' reference measure. It is therefore relatively 'rare' for a Markov semigroup on an infinite-dimensional space to have the strong Feller property. (Basically, some rather strong form of invertibility is typically required of the covariance operator of an infinite-dimensional diffusion for it to generate a strong Feller semigroup, see for example [DPEZ95, DPZ96, EH01].) In particular, one does often not expect to have convergence in a total variation norm as in Theorem 0.5.

This suggests that one should look for an alternative 'weak' (i.e. dealing with weak convergence rather than total variation convergence) theory. This can be achieved by making use of the following notion. An increasing sequence d_n of continuous pseudo-metrics on $\mathcal{X}$ is called *totally separating* if one has $d_n(x,y) \nearrow 1$ as $n \to \infty$ for any $x \neq y$. Each of these metrics can be lifted in a natural way to the space of probability measures on $\mathcal{X}$ by

$$d(\mu,\nu) = \sup_{Lip_d\,\phi=1} \int_{\mathcal{X}} \phi(x)\big(\mu-\nu\big)(dx) \;,$$

where we denote by $Lip_d\,\phi$ the (minimal) Lipschitz constant of $\phi\colon \mathcal{X} \to \mathbf{R}$ with respect to the metric d. With this definition, it turns out [HM06] that for any system of totally separating pseudo-metrics, one has the identity

$$\|\mu - \nu\|_{\mathrm{TV}} = 2\sup_{n\geq 0} d_n(\mu,\nu) \;.$$

On the other hand, it is known [Sei01, Hai08] that the strong Feller property is equivalent to the continuity of transition probabilities in the total variation distance. Therefore, a Markov semigroup is strong Feller if and only if there exists $t \geq 0$ such that, for every $x \in \mathcal{X}$, one has

$$\inf_{U\ni x}\sup_{y\in U}\sup_{n\geq 0} d_n(\mathcal{P}_t(x,\cdot\,),\mathcal{P}_t(y,\cdot\,)) = 0 \;,$$

where the first infimum runs over all neighbourhoods of x. This motivates the following natural extension to the strong Feller property [HM06]:

Definition 0.6. *A Markov semigroup $\mathcal{P}_t$ satisfies the asymptotic strong Feller property if there exists a sequence $t_n \nearrow \infty$ and a system d_n of totally separating* *continuous pseudo-metrics such that, for every $x \in \mathcal{X}$, one has*

$$\inf_{U \ni x} \sup_{y \in U} \sup_{n \geq 0} d_n(\mathcal{P}_{t_n}(x, \cdot), \mathcal{P}_{t_n}(y, \cdot)) = 0 \ .$$

This definition is not only natural, it is also useful as can be seen by [HM06]:

Theorem 0.7. *If a Markov semigroup is asymptotically strong Feller and has an accessible point, then it can have at most one invariant probability measure.*

Under slightly stronger assumption, it is also possible to give a generalisation of Harris' theorem on exponential convergence to this setting [HM08b]. To conclude, let us point out that the asymptotic strong Feller property can be verified in a number of situations where the strong Feller property is either known to fail (stochastic delay equations with fixed delay in the diffusion coefficient [HM08b]) or conjectured to fail (stochastic PDEs satisfying a Hörmander-type condition, but driven only by finitely many Wiener processes [HM08b]).

References

[DPEZ95] G. Da Prato, K. D. Elworthy, and J. Zabczyk. Strong Feller property for stochastic semilinear equations. *Stochastic Anal. Appl.* **13**, no. 1, (1995), 35–45.

[DPZ96] G. Da Prato and J. Zabczyk. *Ergodicity for Infinite Dimensional Systems*, vol. 229 of *London Mathematical Society Lecture Note Series*. University Press, Cambridge, 1996.

[EH01] J.-P. Eckmann and M. Hairer. Uniqueness of the invariant measure for a stochastic PDE driven by degenerate noise. *Commun. Math. Phys.* **219**, no. 3, (2001), 523–565.

[Hai08] M. Hairer. Ergodic properties of a class of non-Markovian processes, 2008. To appear in *Trends in Stochastic Analysis*, Cambridge University Press.

[HM06] M. Hairer and J. C. Mattingly. Ergodicity of the 2D Navier-Stokes equations with degenerate stochastic forcing. *Ann. of Math. (2)* **164**, no. 3, (2006), 993–1032.

[HM08a] M. Hairer and J. Mattingly. Spectral gaps in Wasserstein distances and the 2D stochastic Navier-Stokes equations, 2008. To be published in Ann. Probab.

[HM08b] M. Hairer and J. Mattingly. Yet another look at Harris ergodic theorem for Markov chains, 2008. Preprint.

[HM08b] M. Hairer and J. Mattingly. A Theory of Hypoellipticity and Unique Ergodicity for Semilinear Stochastic PDEs, 2008. Preprint.

[HM08b] M. Hairer, J. Mattingly, and M. Scheutzow. A weak form of Harris theorem with applications to stochastic delay equations, 2008. Preprint.

[MT93] S. P. Meyn and R. L. Tweedie. *Markov Chains and Stochastic Stability*. Springer-Verlag, 1993.

[Sei01] J. Seidler. A note on the strong Feller property, 2001. Unpublished lecture notes.