COUPLING STOCHASTIC PDES

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We consider a class of parabolic stochastic PDEs driven by white noise in time, and we are interested in showing ergodicity for some cases where the noise is degenerate, i.e. acts only on part of the equation. In some cases where the standard Strong Feller - Irreducibility argument fails, one can nevertheless implement a coupling construction that ensures uniqueness of the invariant measure. We focus on the example of the complex Ginzburg-Landau equation driven by real space-time white noise.

1. Introduction

In this work, we consider the long-time behaviour of stochastic partial differential equations of the type

$$dX(t) = AX(t)\,dt + F(X)\,dt + Q\,dW(t)\;, \tag{1}$$

where A is the generator of an analytic semigroup on a Hilbert space $\mathcal{H}$, W is a cylindrical Wiener process on $\mathcal{H}$, $Q : \mathcal{H} \to \mathcal{H}$ is a bounded operator, and $F : D(F) \to \mathcal{H}$ is a suitable nonlinearity. We refer to the monograph by Da Prato and Zabczyk [2] for a number of conditions on A, Q and F that ensure the well-posedness of (1), as well as the existence of a unique stochastic flow $\Phi_t(X)$ that yields the solution of (1) at time t with initial condition $X \in \mathcal{H}$.

One associates to (1) a semigroup $\mathcal{P}_t$ acting on bounded measurable functions $\varphi : \mathcal{H} \to \mathbf{R}$, as well as its dual semigroup $\mathcal{P}_t^*$ acting on Borel probability measures μ by

$$\big(\mathcal{P}_t\varphi\big)(X) = \mathbf{E}\big(\varphi(\Phi_t(X))\big)\;, \qquad \big(\mathcal{P}_t^*\mu\big)(A) = \mathbf{E}\big(\mu(\Phi_t^{-1}(A))\big)\;.$$

From an intuitive point of view, $\mathcal{P}_t\varphi$ describes the evolution of the observable φ, whereas $\mathcal{P}_t^*\mu$ describes the evolution of a distribution of initial conditions. An *invariant measure* for (1) is a probability measure $\mu_\star$ on $\mathcal{H}$ satisfying $\mathcal{P}_t^*\mu_\star = \mu_\star$ for every $t \geq 0$. We do not wish to deal with the question of the existence of an invariant measure in this paper. We therefore take it for granted that the dissipativity properties of A and of F are strong enough to provide tightness for the ergodic averages

$$R_T\mu = \frac{1}{T}\int_0^T \mathcal{P}_t^*\mu\,dt\;,$$

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and thus guarantee the existence of an invariant measure (since every accumulation point of $R_T\mu$ is an invariant measure under some minimal regularity assumptions, see e.g. [3]).

The present paper focuses on the question of the *uniqueness* of the invariant measure for (1). We start in Section 2 by giving a very short review of two of the main methods used to tackle this question. We then proceed in Section 3 by describing how several kinds of coupling methods have recently been applied to this problem. We conclude by explaining in detail how to apply some of the results explained in Section 3 to the complex Ginzburg-Landau equation driven by real-valued space-time white noise.

2. Two general methods

Until recently, two general methods dominated the literature about ergodicity results for the type of parabolic SPDEs considered here. We refer to these two methods as the "dissipativity method" and the "overlap method". The reader interested in a more detailed overview of these two methods is referred to the excellent review paper by Maslowski and Seidler [14], which also contains a more complete list of references.

2.1. *The dissipativity method*

In its most crude form, this method assumes that A and F satisfy the dissipativity condition

$$\big\langle X-Y, A(X-Y)+F(X)-F(Y)\big\rangle \le -c\|X-Y\|^2\;, \tag{2}$$

for some positive constant c and for all X and Y in the domain of A. Under further (rather weak) regularity assumptions on F, (2) implies that any two solutions $X(t)$ and $Y(t)$ of (1) driven by the same realisation of the noise process W converge exponentially toward each other. This immediately implies that if $\mu_\star$ is an invariant measure for (1) and μ is any measure on $\mathcal{H}$ with sufficiently good decay properties, one has $\mathcal{P}_t^*\mu \to \mu_\star$ in the topology of weak convergence.

Of course, many variants of this method are available, in particular one may wish to measure the distance between solutions by a Lyapunov function V which is different from $\|\cdot\|^2$ and more adapted to the problem at hand [13]. It is also possible to formulate conditions analogous to (2) that imply exponential convergence in the case where $\mathcal{H}$ has only a Banach space structure [2]. Unfortunately, there seems to be no way of applying the dissipativity method to situations where the deterministic part of the equation induces chaotic behaviour of the solutions, which is the situation of interest in the study of most problems related to turbulence.

2.2. *The overlap method*

The "overlap method" is based on the following classical theorem by Doob [5]:

Theorem 2.1. *Let $\mathcal{P}_t$ be a Markov semigroup which is irreducible and has the Strong Feller property. Then $\mathcal{P}_t$ admits at most one invariant measure.*

Recall that the strong Feller property means that the semigroup maps bounded measurable functions into continuous functions. Irreducibility means that $\big(\mathcal{P}_t^*\mu\big)(A) > 0$ for every

μ, every $t > 0$, and every non-empty open set A. The conditions of Theorem 2.1 imply that $\mathcal{P}_t^*\mu$ and $\mathcal{P}_t^*\nu$ have a non-zero "overlap" for any two probability measures μ and ν, i.e. there exists a positive measure δ such that $\mathcal{P}_t^*\mu - \delta$ and $\mathcal{P}_t^*\nu - \delta$ are both positive measures. Combining this with the well-known fact from ergodic theory that if $\mathcal{P}_t$ admits more than one invariant measure, at least two of them must be mutually singular yields the result. If one furthermore assumes some bounds on the dissipativity of the equation (in a much weaker sense than in the previous subsection, one basically needs to get bounds on the hitting time of some compact set), the techniques exposed in the monograph of Meyn and Tweedie [17] allow to translate them into bounds on the convergence of an arbitrary initial measure toward the invariant measure. This convergence then takes place in the total variation distance, which can be defined between two measures μ_1 and μ_2 on $\mathcal{H}$ by

$$\|\mu_1 - \mu_2\|_{\mathrm{TV}} = \inf_{\mu \in \mathcal{C}(\mu_1,\mu_2)} \mu\bigl(\mathcal{H}^2 \setminus \bigl\{(x,x) \mid x \in \mathcal{H}\bigr\}\bigr) , \tag{3}$$

where $\mathcal{C}(\mu_1,\mu_2)$ is the set of measures on $\mathcal{H}^2$ with marginals μ_1 and μ_2. Notice that this distance does not take into account the topology of the space $\mathcal{H}$. One can interpret it as measuring the maximal probability for two random variables with respective laws μ_1 and μ_2 to be equal.

For a finite-dimensional SDE with smooth coefficients, the Strong Feller property is a consequence of the hypoellipticity of the operator $\partial_t + L$, where L is the generator of the Markov process. A very efficient criteria for hypoellipticity is given by Hörmander's theorem [10]. However, no satisfactory formulation of Hörmander's theorem is available yet in the infinite-dimensional setting, so the Strong Feller property is usually proved by other means there.

One efficient tool for proving that the strong Feller property holds for an infinite-dimensional system is given by the Bismut–Elworthy–Li formula [8]. In one of its formulations, this formula is given by

$$\bigl(D\mathcal{P}_t\varphi\bigr)(X)h = \frac{2}{t}\,\mathbf{E}\Bigl(\bigl(\varphi\circ\Phi_t\bigr)(X)\int_{t/2}^t \bigl\langle Q^{-1}\bigl(D\Phi_s\bigr)(X)h, dW(s)\bigr\rangle\Bigr) .$$

Here, the notation $(Df)h$ is used to denote the directional derivative of the function f in the direction h. The main feature of this formula is that it yields bounds on $D\mathcal{P}_t\varphi$ that are independent of $D\varphi$. However, we notice that it requires the range of $D\Phi_s$ to be smaller than the domain of Q^{-1} in order to be applicable. This condition can be verified in many important particular cases [3, 1], but it usually requires the kernel of Q to be $\{0\}$.

In the linear case (i.e. when $F = 0$), a complete understanding of the conditions implying the strong Feller property is available [2, 19]. In particular, it is known that the strong Feller property is then equivalent to the exact null controllability of the control system $\dot X = AX + Qu$, with $u \in L^2([0,t],\mathcal{H})$. This allows for covariance operators Q that are very degenerate. It intuitively agrees with the definition of the total variation distance, since it means that under a suitable change of measure (given by the Girsanov transform $dW(t) \mapsto dW(t) + u\,dt$), two solutions with different initial conditions will meet after some time.

Unfortunately, no such characterisation of the strong Feller property is available in the non-linear case, the closest approximation to it being the coupling method described in the next section. However, if the range of the nonlinearity F is contained in the range of Q, the strong Feller property can in some cases be recovered by performing a Girsanov transform to eliminate the non-linearity [15]. In some cases, it is also possible to recover the strong Feller property by adapting Malliavin's proof of Hörmander's theorem to the infinite-dimensional case, but to the knowledge of the author, only the particular case of a reaction-diffusion type equation with a covariance operator Q having a kernel of finite co-dimension has been treated by this method so far [7].

3. The coupling method

In this section, we present one way to apply coupling techniques to the problem of ergodicity for stochastic PDEs. Recall that a *coupling* for a pair of measures μ_1 and μ_2 is a measure μ on the product space with marginals μ_1 and μ_2. In the context of stochastic PDEs, we call a coupling for (1) a family of stochastic processes $(X(t), Y(t))$ indexed by their initial conditions $(X(0), Y(0)) \in \mathcal{H}^2$ and such that $X(t)$ and $Y(t)$ both solve (1). The basic idea of any coupling method is to introduce correlations between $X(t)$ and $Y(t)$ in such a way that $\|X(t) - Y(t)\| \to 0$ as $t \to \infty$. The precise type and speed of this convergence then determines the topology and the speed of convergence of $\mathcal{P}_t^*\mu$ to the (unique) invariant measure $\mu_\star$. Although coupling methods have been used to prove ergodicity results for stochastic evolutions since the late thirties (see e.g. [4]), they seem to have been applied successfully to stochastic PDEs only recently.

Actually, both the dissipativity and the overlap method can be interpreted as special cases of couplings. In the dissipativity method, one drives $X(t)$ and $Y(t)$ with the same realisation of the noise process and relies on the dissipativity of the equation to drive both processes toward each other. In the overlap method, one first discretises time (by looking at integer times, say) and then uses the maximal coupling for the transition probabilities. The transition probabilities $P(X, Y, \cdot\,)$ for the coupled process are thus given by the coupling that realises the infimum in (3) with $\mu_1 = P(X, \cdot)$ and $\mu_2 = P(Y, \cdot)$. This coupling can easily be shown to exist and to be unique. (Here, P denotes the time 1 transition probabilities for (1).) In other words, the maximal coupling will try as hard as it can to force $X(t)$ and $Y(t)$ to become equal and, once it succeeds, they will be guaranteed to remain equal for all subsequent times. If one starts X in the invariant measure $\mu_\star$ and Y in an arbitrary measure μ, one has $\|\mathcal{P}_t^*\mu - \mu_\star\|_{\mathrm{TV}} \le \mathbf{P}\{\tau > t\}$, where τ is the first time at which $X(\tau) = Y(\tau)$.

In a series of recent papers, E, Mattingly, Sinai [6, 16] and Kuksin, Shirikyan [12, 11] realised that, loosely speaking, it is possible in certain cases to split the state space of (1) into two spaces $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and to combine the dissipativity method on $\mathcal{H}_-$ with the overlap method on $\mathcal{H}_+$. Let us call $\mathcal{H}_+$ the "unstable modes" and $\mathcal{H}_-$ the "stable modes" and assume that $\mathcal{H}_+$ is finite-dimensional. In both cases, the authors focused on the example of the 2D Navier-Stokes equation on a torus with periodic boundary conditions, forced by a noise with a covariance operator Q that has the important property that $\Pi_+ Q \Pi_+$ is invertible on $\mathcal{H}_+$. (We denote by $\Pi_\pm$ the orthogonal projections on $\mathcal{H}_\pm$.) Under this condition, they constructed a coupling with the following properties. On $\mathcal{H}_+$, it behaves like the maximal coupling, i.e. it maximises the probability for the $\mathcal{H}_+$ components $X_+(t)$ and $Y_+(t)$ to become equal. On $\mathcal{H}_-$ on the other hand, it drives X_- and Y_- with identical realisations of that component of the noise process. The space $\mathcal{H}_-$ is chosen in such a way that the dynamic then tends to steer X_- and Y_- toward each other. (In the typical situation of a semilinear parabolic equation, the space $\mathcal{H}_-$ would consist of Fourier modes with sufficiently high wave number.)

The main difficulty of this approach comes from the fact that, unlike in the situation of the overlap method, having $X_+(t) = Y_+(t)$ at a certain time does not allow one to ensure that $X_+(s) = Y_+(s)$ for all subsequent times $s > t$. The reason is that one has in general $X_-(t) \neq Y_-(t)$ and therefore, if one drives X and Y with the *same* realisation of the noise, the influence of the stable modes will tend to drive the unstable modes away from each other. Call W_x and W_y the noises driving X and Y respectively. The (finite-dimensional) processes X_+ and Y_+ are then solutions of SDEs of the following type:

$$dX_+(t) = g(X_+, X_-)\,dt + \Pi_+ Q \Pi_+\,dW_x(t)\;,$$
$$dY_+(t) = g(Y_+, Y_-)\,dt + \Pi_+ Q \Pi_+\,dW_y(t)\;.$$

Since $\Pi_+ Q \Pi_+$ is invertible by assumption, one notices that the Girsanov transform

$$dW_y(t) = dW_x(t) + (\Pi_+ Q \Pi_+)^{-1}\big(g(X_+, X_-) - g(Y_+, Y_-)\big)\,dt\;, \tag{4}$$

allows to have $X_+(t) = Y_+(t)$ for all times. Of course, dW_y as defined by (4) is not a Wiener process anymore, so this is not an acceptable coupling. It is nevertheless possible to construct a coupling that gives positive mass to the event (4). Furthermore, as long as it is satisfied, the difference $\|X_- - Y_-\|$ converges to 0, so the difference between dW_x and dW_y in (4) becomes smaller and smaller. The coupling can be constructed in such a way that it therefore becomes more and more likely for the event (4) to be satisfied. By carefully estimating these probabilities, one can show that the random time $\tau = \inf\{t > 0 \,|\, X_+(s) = Y_+(s)\, \forall s > t\}$ is almost surely finite. Estimates on τ and on the speed at which $\|X_- - Y_-\| \to 0$ then immediately translate into estimates on the speed at which $\mathcal{P}_t^*\mu$ converges to the invariant measure.

Another method for constructing couplings for an equation of the type (1) is to construct two operator-valued functions G_1 and G_2 such that

$$G_1 G_1^* + G_2 G_2^* = \text{Identity}\;, \tag{5}$$

and to consider the couple of equations

$$dX(t) = AX(t)\,dt + F(X)\,dt + QG_1(X,Y)\,dW_1(t) + QG_2(X,Y)\,dW_2(t)\;,$$
$$dY(t) = AY(t)\,dt + F(Y)\,dt + QG_1(X,Y)\,dW_1(t) - QG_2(X,Y)\,dW_2(t)\;,$$

where W_1 and W_2 are two independent cylindrical Wiener processes. It is clear that this is a coupling for (1) for any choice of the G_i satisfying (5). It was shown in [18] that, for a certain class of semilinear parabolic equations, it is possible to choose the G_i in such a way that the stopping time $\tau = \inf\{t > 0 \,|\, X(t) = Y(t)\}$ is almost surely finite, therefore obtaining convergence toward the invariant measure in the total variation distance.

We finally turn to the coupling technique developed in [9]. This technique is very close in spirit to the one exposed in [16, 12, 11]. However it allows in some cases to treat a situation where the noise acts in a degenerate way on the unstable part of the equation. The idea exposed in [9] is to look for a Banach space $\mathcal{B} \subset \mathcal{H}$ with norm $\|\cdot\|_\star$ and for a function $G : \mathcal{B}^2 \to \mathcal{H}$ with the following properties:

P1 The solutions of (1) are almost-surely $\mathcal{B}$-valued and they satisfy

$$\mathbf{E}\|u(t)\|_\star^p \le C(t)(1 + \|u(0)\|_\star^p) \;.$$

P2 The function G is bounded by $\|G(u,v)\| \le C\|u - v\|^\varepsilon (1 + \|u\|_\star + \|v\|_\star)^N$ for some positive constants C, ε, N, and for all u and v in $\mathcal{B}$.

P3 If u is the solution of (1) driven by dW and v is the solution of (1) driven by $dW' = dW + G(u,v)\,dt$, then $\|u(t) - v(t)\|_\star \to 0$ exponentially as $t \to \infty$.

The main result of [9] is then:

Theorem 3.1. *Consider (1) and assume that there exists a function G and a Banach space $\mathcal{B} \subset \mathcal{H}$ satisfying **P1**–**P3**. Then, (1) has a unique invariant measure and, for every initial condition in $\mathcal{H}$, the solution converges to the invariant measure at an exponential rate.*

Remark 3.1. The convergence takes place in a Wasserstein distance which is the same as the one usually used in the dissipativity method. The topology defined by this distance is slightly stronger than the topology of weak convergence.

There does not seem to be a general recipe for finding a suitable function G. On the other hand, when the nonlinearity is small (not in the sense of "arbitrarily small" like in perturbation theory, but in the sense that the noise gets transmitted to the whole phase space through the linear part of the equation and that the nonlinearity does not affect this behaviour), we will see in the following section that it is not too difficult to find a good function G.

4. The complex Ginzburg-Landau equation

We now focus on the following equation, which is a stochastic perturbation of the complex Ginzburg-Landau equation:

$$du(x,t) = \big((1+i\alpha)\Delta u(x,t) + (1+i\beta)u(x,t) - u(x,t)|u(x,t)|^2\big)\,dt + dW(x,t) \;. \qquad \text{(CGL)}$$

In this equation, α and β are two real-valued parameters, $x \in [-L, L]$, Δ denotes the Laplace operator, and $u(\cdot, t)$ is assumed to satisfy periodic boundary conditions. Furthermore, $\frac{dW}{dt}(x,t)$ formally denotes space-time white noise, i.e. it is a distribution-valued Gaussian process with covariance

$$\mathbf{E}\Big(\frac{dW}{dt}(x,t)\frac{dW}{dt}(y,s)\Big) = \delta(x-y)\delta(t-s) \;.$$

It is a standard result (see [2]) that (CGL) has a unique $\mathcal{H} = L^2([-L,L])$-valued solution. We denote again by $\mathcal{P}_t$ and $\mathcal{P}_t^*$ the semigroups associated to it. The remainder of this paper is devoted to the proof of the following result:

Theorem 4.1. *For every pair (α, β) satisfying the conditions*

$$|\beta| \ge \frac{3}{2}\;, \qquad \alpha^2 \le \frac{24\beta^2}{36\beta^2 + (15-\beta^2)^2}\;, \qquad (6)$$

the equation (CGL) has a unique invariant measure and all solutions converge exponentially to it.

Remark 4.1. The conditions on α and on β are not sharp, not even within the framework of the method of proof presented here.

Proof. All the calculations in this proof will be performed at a formal level. A rigorous justification is relatively easy by using the regularising properties of the equation (CGL). It is indeed a standard result that the solutions of (CGL) are almost surely continuous functions of space and of time. In particular, condition **P1** holds for $\mathcal{B}$ equal to the space of continuous functions equipped with the supremum norm.

Denoting the real and imaginary parts of a complex-valued function u by u_r and u_i, (CGL) becomes

$$\begin{aligned} du_r(x,t) &= \big(\Delta u_r - \alpha\Delta u_i + u_r - \beta u_i - u_r(u_r^2+u_i^2)\big)\,dt + dW(x,t)\;, \\ du_i(x,t) &= \big(\Delta u_i + \alpha\Delta u_r + u_i + \beta u_r - u_i(u_r^2+u_i^2)\big)\,dt\;. \end{aligned}$$

Our aim is to give an explicit expression for a function G satisfying properties **P1**–**P3** of the previous section. Denote by $v = v_r + i\,v_i$ the solution of (CGL) driven by the noise process $dW' = dW + G\,dt$ and define $\varrho = v - u$. One then has for ϱ:

$$d\varrho_r = \big(\Delta\varrho_r - \alpha\Delta\varrho_i + \varrho_r - \beta\varrho_i - v_r(v_r^2+v_i^2) + u_r(u_r^2+u_i^2)\big)\,dt + G(u,v)\;, \tag{7a}$$

$$d\varrho_i = \big(\Delta\varrho_i + \alpha\Delta\varrho_r + \varrho_i + \beta\varrho_r - v_i(v_r^2+v_i^2) + u_i(u_r^2+u_i^2)\big)\,dt\;. \tag{7b}$$

The most "dangerous" part of this equation is the linear instability. Before we proceed, we therefore consider the following simplified equations:

$$\frac{d\tilde\varrho_r}{dt} = \tilde\varrho_r - \beta\tilde\varrho_i + \tilde G\;, \qquad \frac{d\tilde\varrho_i}{dt} = \tilde\varrho_i + \beta\tilde\varrho_r\;. \tag{8}$$

Let us introduce the variable $\xi = \beta\tilde\varrho_r + 3\tilde\varrho_i$ and rewrite the second equation as

$$\frac{d\tilde\varrho_i}{dt} = -2\tilde\varrho_i + \xi\;. \tag{9}$$

Choosing

$$\tilde G = -5\tilde\varrho_r + \Big(\beta - \frac{6}{\beta}\Big)\tilde\varrho_i\;, \tag{10}$$

we obtain for ξ the equation $\frac{d}{dt}\xi = -\xi$. Therefore, ξ converges exponentially to 0, which in turn implies by (9) that ϱ_i converges exponentially to 0. Since ϱ_r is a linear combination of ξ and ϱ_i, it converges exponentially to 0 as well. To characterise this convergence more precisely, we introduce the norm

$$|\!|\!|\tilde\varrho|\!|\!|^2 = \tilde\varrho_r^2 + \frac{15}{\beta^2}\tilde\varrho_i^2 + \frac{6}{\beta}\tilde\varrho_r\tilde\varrho_i\;. \tag{11}$$

A straightforward computation shows that, with the choice (10) for $\tilde G$, one has

$$\frac{d|\!|\!|\tilde\varrho|\!|\!|^2}{dt} = -2\tilde\varrho_r^2 - \frac{6}{\beta^2}\tilde\varrho_i^2 < -\gamma|\!|\!|\tilde\varrho|\!|\!|^2\;,$$

for some constant $\gamma > 0$.

We now turn to the full equation (7), which we rewrite as

$$\frac{d\varrho_r}{dt} = \Delta\varrho_r - \alpha\Delta\varrho_i - 4\varrho_r - \frac{6}{\beta}\varrho_i + G_2(u,v)\;, \tag{12a}$$

$$\frac{d\varrho_i}{dt} = \Delta\varrho_i + \alpha\Delta\varrho_r + \varrho_i + \beta\varrho_r - (u_i + \varrho_i)\big(u_r\varrho_r + u_i\varrho_i + \varrho_r^2 + \varrho_i^2\big) - \varrho_i(u_r^2 + u_i^2)\;, \tag{12b}$$

where we defined $G = G_1 + G_2$ with

$$G_1 = -5\tilde\varrho_r + \Big(\beta - \frac{6}{\beta}\Big)\tilde\varrho_i + v_r(v_r^2 + v_i^2) - u_r(u_r^2 + u_i^2)\;.$$

Notice that G_1 satisfies **P1**–**P2** with $\|\cdot\|_\star$ replaced by the supremum norm. By analogy with (11), we introduce the norm

$$|\!|\!|\tilde\varrho|\!|\!|^2 = \|\tilde\varrho_r\|^2 + a\|\tilde\varrho_i\|^2 + b\langle\tilde\varrho_r,\tilde\varrho_i\rangle\;,\qquad a = \frac{15}{\beta^2}\;,\quad b = \frac{6}{\beta}\;,$$

so that

$$\frac{d|\!|\!|\tilde\varrho|\!|\!|^2}{dt} \le -\gamma|\!|\!|\varrho|\!|\!|^2 - T_1 - T_2\;,$$

where we defined

$$T_1 = (2+\alpha b)\|\nabla\varrho_r\|^2 + (2a - \alpha b)\|\nabla\varrho_i\|^2 + 2(b + a\alpha - \alpha)\langle\nabla\varrho_r,\nabla\varrho_i\rangle\;,$$

and

$$T_2 = -\langle 2\varrho_r + b\varrho_i, G_2\rangle + \big\langle 2a\varrho_i + b\varrho_r, (u_i + \varrho_i)\big(u_r\varrho_r + u_i\varrho_i + \varrho_r^2 + \varrho_i^2\big) + \varrho_i(u_r^2 + u_i^2)\big\rangle\;.$$

Due to the second condition in (6), T_1 is always positive. It thus remains to find a function G_2 satisfying **P1**–**P2** and such that T_2 is always positive.

Since G_2 is multiplied by $2\varrho_r + b\varrho_i$, we can choose it in such a way to replace every occurrence of ϱ_r by $-\frac{b}{2}\varrho_i$ in the above expression, thus yielding

$$T_2 = \frac{4a - b^2}{3}\Big\langle\varrho_i, \Big(u_r^2 + 2u_i^2 - \frac{b}{2}u_iu_r + \big(2 + \frac{b^2}{4}\big)\varrho_iu_i + \big(1 + \frac{b^2}{4}\big)\varrho_i^2 - \frac{b}{2}u_r\varrho_i\Big)\varrho_i\Big\rangle\;.$$

Since $4a - b^2$ is positive, positivity of T_2 is equivalent to the matrix

$$\Gamma = \frac{1}{4}\begin{pmatrix} 4 & -b & -b \\ -b & 8 & 4 + \frac{b^2}{2} \\ -b & 4 + \frac{b^2}{2} & 4 + b^2\end{pmatrix}$$

being positive definite. Since Γ is positive definite for $b = 0$, it remains so until the first value of b for which

$$\det\Gamma = \frac{1}{64}\big(64 + 12b^2 - b^4\big) = 0\;.$$

Therefore, under the condition $|b| \le 4$, (i.e. $|\beta| > \frac{3}{2}$), one has $\frac{d}{dt}|\!|\!|\tilde\varrho|\!|\!|^2 \le -\gamma|\!|\!|\varrho|\!|\!|^2$ almost surely. Applying Theorem 3.1 concludes the proof of Theorem 4.1. □

Remark 4.2. Note that the same proof goes through if one multiplies the cubic term in (CGL) by a factor $(1 + i\gamma)$ with $\gamma \in \mathbf{R}$ small enough.

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