Ergodic Properties of Markov Processes

Exercises for week 3

Exercise 1 Check that $P^{n+m}(a, A) = \int_{\mathcal{X}} P^n(x, A)\, P^m(a, dx)$ for every $n, m \geq 1$ and that the operator T^n defined by $(T\mu)(A) = \int_{\mathcal{X}} P^n(x, A)\, \mu(dx)$ is equal to the operator obtained by applying T n times, $T^n = T \circ T \circ \ldots \circ T$.

Exercise 2 Let ξ_n be a sequence of real-valued i.i.d. random variables and define x_n recursively by $x_0 = 0$, $x_{n+1} = \alpha x_n + \xi_n$. This process is called the **autoregressive process**.

Show that this is a time-homogeneous Markov process and write its transition probabilities in the cases where (1) the ξ_n are Bernoulli random variables (*i.e.* $\xi_n = 0$ with probability $1/2$ and $\xi_n = 1$ otherwise) and (2) the law of ξ_n has a density p with respect to the Lebesgue measure on **R**.

In the case (1) with $\alpha < 1/2$ (say $\alpha = 1/3$), what does the law of x_n look like for large values of n?

Exercise 3 Let $\{\xi_n\}$ be a sequence of i.i.d. random variables that take values in $\{-1, 1\}$ with equal probabilities and define recursively $x_0 = 0$, $x_{n+1} = x_n + \xi_n$. Which of the following random times are stopping times?

- $T_0 = \inf\{n \geq 4$ such that x_n is even$\}$.
- $T_1 = T_0 - 1$, $T_2 = T_0 - 2$, $T_3 = T_0 + 1$.
- $T_4 = \inf\{n \geq 0$ such that $x_{n+5} \geq x_n + 2\}$.
- $T_4 = \inf\{n \geq 5$ such that $x_{n-5} \geq x_n + 2\}$.

Exercise 4 Show that if the state space $\mathcal{X}$ is countable and T is an arbitrary linear operator on the space of finite signed measures which maps probability measures into probability measures, then T is of the form $T\mu(A) = \int_{\mathcal{X}} P(x, A)\, \mu(dx)$ for some P.

Hint Apply T to delta-measures.

* **Exercise 5** Let $\{\xi_n\}$ be a sequence of i.i.d. random variables that take values in $\{1, 2, 3, 4\}$ with equal probabilities and define $a_0 = 0$, $a_n = \sum_{i=1}^n \xi_i$. Set $x_0 = 0$ and, for $n \geq 1$,

$$x_n = n - \sup\{a_j \mid a_j \leq n\} .$$

Draw a picture corresponding to this situation and use it to convince yourself that one has $x_n \in \{0, 1, 2, 3\}$ for every n. Show that x_n is a Markov process and write down its transition probabilities.

* **Exercise 6** Show that a process is Markov if and only if, for every $n > 0$, one has

$$\mathbf{P}(x_n \in A \mid x_0 = a_0, \ldots, x_{n-1} = a_{n-1}) = \mathbf{P}(x_n \in A \mid x_{n-1} = a_{n-1}) ,$$

for every set $A \in \mathscr{B}(\mathcal{X})$ and almost every sequence $(a_0, \ldots, a_{n-1}) \in \mathcal{X}^n$.

Hint Use the fact that this property is equivalent to $\mathbf{E}(f(x_n) \mid \mathscr{F}_0^{n-1}) = \mathbf{E}(f(x_n) \mid \mathscr{F}_{n-1})$ for every f and every n.

* **Exercise 7** Let x be a Markov process on $\mathcal{X}$, let $A \subset \mathcal{X}$, and define a sequence of stopping times by $T_{-1} = -1$ and $T_n = \inf\{k \geq T_{n-1} \,|\, x_k \in A\}$. Define a process y on $A \cup \{\star\}$ by

$$y_n = \begin{cases} x_{T_n} & \text{if } T_n < \infty, \\ \star & \text{otherwise.} \end{cases}$$

Show that y is again a Markov process and relate its transition probabilities to those of x.

** **Exercise 8** Show that the autoregressive process is always Feller and that it is strong Feller if the law of ξ_n has a density with respect to the Lebesgue measure. Show that in the case where the ξ_n are Bernoulli random variables, it is *not* strong Feller.

** **Exercise 9** Show that the conclusions of Exercise 4 still hold under the assumptions that $\mathcal{X}$ is a complete separable metric space and T is continuous in the weak topology.

Hint Prove first that with these assumptions, every probability measure can be approximated in the weak topology by a finite sum of δ-measures (with some weigths). Remember that a space is called separable if it contains a countable dense subset.