Φ^4_3 is orthogonal to GFF

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Abstract

We show that the finite-volume Φ^4 measure on the 3-dimensional torus is singular with respect to the Gaussian Free Field (GFF) measure. This is in contrast to the two-dimensional case where the corresponding measures are equivalent, see [Nel66]. The proof furthermore reveals that the Φ^4_3 measure fails to be quasi-invariant under any smooth non-zero shift.

1 The proof

Take $\varepsilon_n = \exp(-e^n)$ and, for any distribution Φ as well as $n \geq 0$ write

$$\Phi_n = \Phi \star \varrho_n \;, \qquad \varrho_n(x) = \varepsilon_n^{-d} \varrho(x/\varepsilon_n) \;,$$

for some nice ϱ integrating to 1. For any fixed test function ψ, we then consider the event

$$A_\psi = \Big\{ \Phi \,:\, \lim_{n\to\infty} e^{-3n/4} \Big\langle \Phi_n^3 - 3a e^{e^n} \Phi_n - 9b e^n \Phi, \psi \Big\rangle = 0 \Big\} \;, \tag{1.1}$$

for suitably chosen a and b. We claim that

Theorem 1.1 *Let μ be the Φ^4_3 measure and μ_0 be the GFF. Then, there exist choices of a and b such that $\mu(A_\psi) = 1$ for every smooth ψ. On the other hand, $\mu_0(A_\psi) = 0$ for every choice of a, b with $b \neq 0$.*

Furthermore, for any smooth $\hat\psi \not\equiv 0$, and for a and b as above, there exists a smooth test function ψ such that one has $\mu(A_\psi + \hat\psi) = 0$.

Remark 1.2 The $3/4$ appearing in the exponential could be anything between $1/2$ and 1. At $1/2$, the limit does not exist anymore under μ, but the corresponding sequence is tight. If we make the exponent greater than 1, then the event fails to distinguish between μ and μ_0.

Remark 1.3 The fact that the last occurrence of Φ in (1.1) does not have a subscript n is not a typo, even though this would probably not make much of a difference. It turns out however that the analysis is more natural as it is written.

Remark 1.4 At least formally, the lack of quasi-invariance under shifts by smooth functions is closely related to the fact that, if Φ is a sample from either μ (or from μ_0 for that matter), then there is no good candidate for its "renormalised cube". Some rigorous versions of this fact were known before, see for example [ALZ06] for an "infinitesimal version" of this lack of quasi-invariance.

The fact that $\mu_0(A_\psi) = 0$ is easy to show, so we only consider the case of the Φ^4_3 measure. We exploit the fact that this measure is invariant under the Φ^4_3 dynamics [HM15], so that we can sample from it by looking at the solution at some fixed time (say 1) with initial condition μ. It follows from [Hai14, Sec. 10] that this solution can be written as a modelled distribution in $\mathscr{D}^\gamma$ for $\gamma \in (1, 3/2)$ as

$$\mathbf{\Phi} = \uparrow + \varphi\,\mathbf{1} + \Psi + 3\varphi Y + \varphi'\,X\;,$$

for some continuous functions φ and φ'. (Actually, φ is almost Hölder-$1/2$ and the same is true for φ' but we don't care about that. Also, φ' is of course *not* the derivative / gradient of φ...) The underlying model here is the Φ^4_3 model (Π, Γ) constructed in [Hai14]. This is a model on space-time, but its restriction to the relevant sector can be viewed as a model on space for any fixed instant of time, see [HM15]. We write the corresponding distribution as $\Phi = \mathscr{R}\mathbf{\Phi}$ with $\mathscr{R}$ the reconstruction operator, which in this case, is simply given by $\Phi = \Pi_z \uparrow + \varphi$, where $\Pi_z \uparrow$ is a GFF.

Now, an explicit calculation (or alternatively an application of [Hai14, Sec. 4]) shows that Φ_n can again be viewed as an element of $\mathscr{D}^\gamma$ with

$$\mathbf{\Phi}_n = \dot{\uparrow} + \tilde\varphi_n\,\mathbf{1} + \dot\Psi + 3\varphi \dot Y + \tilde\varphi_n'\,X\;,$$

for some functions $\tilde\varphi_n$ and $\tilde\varphi_n'$ as above. Furthermore, the elements $\mathbf{\Phi}_n$ converge in $\mathscr{D}^\gamma$ to a limit $\mathbf{\Phi}_\infty = \dot{\uparrow} + \varphi\,\mathbf{1} + \dot\Psi + 3\varphi \dot Y + \varphi'\,X$.

Note that the coefficient in front of $\dot Y$ is the "old" φ! Note also that one has $\tilde\varphi_n = \varrho_n \star \varphi =: \varphi_n$, but that $\tilde\varphi_n'$ does not in general coincide with either $\varrho_n \star \varphi'$ or with the gradient of φ_n. Here, the red dot appearing in the symbols represents the operation of convolution with ϱ_n, so we are really working in an extended regularity structure. The model (Π, Γ) (now viewed as a model on space only!) is extended canonically to this larger structure to a model $(\Pi^{(n)}, \Gamma^{(n)})$ as in [Hai14, Thm ??]. The reason for the "red dot" notation is to suggest that as $n \to \infty$ the model on the symbols with red dots converges to that with the red dots removed.

It follows that $\mathbf{\Phi}_n^3$ is an element of $\mathscr{D}^{\bar\gamma}$ for some $\bar\gamma > 0$, given by

$$\mathbf{\Phi}_n^3 = \dot\Xi^3 + 3\tilde\varphi_n\,\dot\Xi^2 + 3\,\dot\Xi^2\dot\Psi + 9\varphi\,\dot\Xi^2\dot Y + 3\tilde\varphi_n'\,\dot\Xi^2 X + 3\tilde\varphi_n^2\,\dot{\uparrow} + 6\tilde\varphi_n\,\dot{\uparrow}\dot\Psi + \tilde\varphi_n^3\,\mathbf{1}\;.$$

Now if we canonically extend $(\Pi^{(n)}, \Gamma^{(n)})$ to this structure, we see that things go bad. However, we have the following facts.

Proposition 1.5 *Let M_n be the renormalisation map associating the constants ae^{e^n} to $\vee$ and be^n to $\Upsilon^{\vee\vee}$ as in [BHZ16], so that in particular*

$$\begin{aligned} M_n\vee &= \vee - ae^{e^n}\mathbf{1}\,, & M_n\Upsilon^{\Psi} &= \Upsilon^{\Psi} - ae^{e^n}\Upsilon - 3be^n\uparrow\,,\\ M_n\vee X &= \vee X - ae^{e^n}X\,, & M_n\Upsilon^{\vee\vee} &= \Upsilon^{\vee\vee} - ae^{e^n}\Upsilon - be^n\mathbf{1}\,,\\ M_n\Psi &= \Psi - 3ae^{e^n}\mathrm{I}\,, \end{aligned}$$

and denote by $(\hat\Pi^{(n)}, \hat\Gamma^{(n)})$ the corresponding renormalised model. Then, there exist choices of a and b such that on the sector spanned by all basis vectors except for Ψ, $(\hat\Pi^{(n)}, \hat\Gamma^{(n)})$ converges in probability to a limiting model $(\hat\Pi, \hat\Gamma)$ satisfying

$$\hat\Pi_z\hat\tau = \Pi_z\tau\,,\quad (\hat\tau,\tau) \in \{(\vee, \vee), (\Upsilon^{\Psi}, \Upsilon^{\Psi}), (\Upsilon^{\vee\vee}, \Upsilon^{\vee\vee}), (\Upsilon^{\vee}, \Upsilon^{\vee})\}\,.$$

Furthermore, one has $e^{-\beta n}\hat\Pi^{(n)}\Psi \to 0$ in probability in $\mathcal{C}^\alpha$ for every $\alpha < -\frac{3}{2}$ and every $\beta > \frac{1}{2}$.

Proof. Basically a corollary of [BHZ16], only the last statement needs some verifying. □

Let now $\hat{\mathcal{R}}^{(n)}$ denote the reconstruction operator associated to $(\hat\Pi^{(n)}, \hat\Gamma^{(n)})$. A simple calculation shows that one has

$$\hat{\mathcal{R}}^{(n)}\mathbf{\Phi}_n^3 = \Phi_n^3 - 3ae^{e^n}\Phi_n - 9be^n\Phi\,.$$

Combining this with Proposition 1.5 immediately shows that

$$\Phi_n^3 - 3ae^{e^n}\Phi_n - 9be^n\Phi - \hat\Pi^{(n)}\Psi\,,$$

converges in probability to a limiting distribution as $n \to \infty$. Taking into account the last statement of Proposition 1.5 and the fact that

$$e^n|\langle\Phi_n - \Phi, \psi\rangle| \lesssim e^n\varepsilon_n\|\Phi\|_{\mathcal{C}^{-1}} \to 0\,,$$

implies that $\mu(A_\psi) = 1$.

To show that $\mu(A_\psi + \hat\psi) = 0$, we first note that

$$\mathbf{\Phi}_n^2 = \vee + 2\tilde\varphi_n\,\mathrm{I} + 2\Upsilon^{\vee} + \tilde\varphi_n^2\,\mathbf{1}\,,$$

so that in particular, similarly to above,

$$\hat{\mathcal{R}}^{(n)}\mathbf{\Phi}_n^2 = \Phi_n^2 - ae^{e^n}\,,$$

thus showing in the same way that $\Phi_n^2 - ae^{e^n}$ converges in probability to a limiting distribution. We now write

$$\begin{aligned}(\Phi_n - \hat{\psi})^3 - (3ae^{e^n} + 9be^n)(\Phi_n - \hat{\psi}) = \Phi_n^3 &- 3ae^{e^n}\Phi_n - 9be^n\Phi - \hat{\Pi}^{(n)}\Psi \\ &- 3\hat{\psi}(\Phi_n^2 - ae^{e^n}) + 3\hat{\psi}^2\Phi_n - \hat{\psi}^3 \\ &+ \hat{\Pi}^{(n)}\Psi \\ &+ 9be^n\hat{\psi}\;.\end{aligned}$$

We now note that the first two lines converge to finite limits while the third line converges to 0 when multiplied by $e^{-\beta n}$ and tested against ψ. The term on the last line however gives rise to $9be^{(1-\beta)n}\langle\hat{\psi},\psi\rangle$ which diverges for some choices of ψ.

References

[ALZ06] S. ALBEVERIO, S. LIANG, and B. ZEGARLINSKI. Remark on the integration by parts formula for the φ_3^4-quantum field model. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **9**, no. 1, (2006), 149–154. doi:10.1142/S0219025706002275.

[BHZ16] Y. BRUNED, M. HAIRER, and L. ZAMBOTTI. Algebraic renormalisation of regularity structures. *ArXiv e-prints* (2016). arXiv:1610.08468.

[Hai14] M. HAIRER. A theory of regularity structures. *Invent. Math.* **198**, no. 2, (2014), 269–504. arXiv:1303.5113. doi:10.1007/s00222-014-0505-4.

[HM15] M. HAIRER and K. MATETSKI. Discretisations of rough stochastic PDEs. *ArXiv e-prints* (2015). arXiv:1511.06937. AoP, to appear.

[Nel66] E. NELSON. A quartic interaction in two dimensions. In *Mathematical Theory of Elementary Particles (Proc. Conf., Dedham, Mass., 1965)*, 69–73. M.I.T. Press, Cambridge, Mass., 1966.